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A new approach to the determination of the critical couplings for a deconfinement phase transition in the variational-cumulant expansion*

Q. Lu[1,a], H. Li[1], T. Chen[2]

[1] Department of Physics, Nankai University, Tianjin 300071, P.R. China
[2] CCAST (World Laboratory) P.O. Box 8730, 10080 Beijing, P.R. China

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Abstract. Based on the variational-cumulant expansion (VCE), a new approach is adopted to determine the critical couplings for the deconfinement phase transition in SU(2) gauge theory for $N_\tau = 2, 3, 4$, and with both the standard Wilson action and the improved tree-level Symansik action. New results of the VCE which are close to the data of Monte Carlo (MC) simulations make manifest that the new approach is much more effective than the traditional one and show the consistence of a VCE analysis with an MC simulation.

1 Introduction

The most powerful technique in order to extract information from QCD in lattice gauge theory is the Monte Carlo approach. This has produced over the past twenty years an imposing amount of results. Yet, in spite of two decades of efforts the mechanisms of QCD are still not clear. It is our belief that in some respects insight in the QCD cannot be achieved without developing, in strict connection with the MC approach, an analytic lattice approach. In this way information coming from the MC can be used in the examination of the analytic approach and be a criterion thereby, and vice versa the analytic analysis is the source of and a complement to MC calculations. For instance, in lattice gauge theory the continuum limit cannot be completely recovered due to system errors. As the lattice spacing $a \to 0$, the cost of a realistic simulation of QCD will grow like some large power, a^{-6} or even a^{-10} [1, 2]. Symanzik's improvement program, as an analytic lattice theory, is designed to systematically reduce the cutoff dependence near the continuum limit and has been proved to be effective. In addition, by making use of a strong-coupling expansion, Polyakov and Susskind have shown that the deconfinement phase transition at a finite temperature for lattice QCD is indicated by the Polyakov line [3]. This analysis was later confirmed by many MC simulations [4–6]; the VCE method as well as the strong-coupling expansion has been confirmed as an effective analytical method both at zero [7,8] and finite temperature [9]. In this paper, based on SU(2)'s similarity to QCD and its simplicity, we present a new method in order to improve the determination of the critical coupling based on the VCE method for the SU(2) gauge model with $N_\tau = 2, 3, 4$. The results, which are rather better than those obtained by the traditional method and which are close to the data of MC, indicate the convergence of the cumulant expansion of the Polyakov line for the SU(2) gauge model and make manifest the consistence of a VCE analysis with an MC calculation.

2 SU(2) lattice gauge models in VCE at finite temperature

Consider the SU(2) lattice gauge theory on a D-dimensional virtual hypercubic lattice, $\Lambda = (N_\tau A_\tau)\times (N_\sigma A_\sigma)^{D-1}$, where $(N_\tau A_\tau)$ is the time-like length while $(N_\sigma A_\sigma)^{D-1}$ is the "space" volume. At finite temperature, we apply the periodic boundary condition in the time-like direction. The temperature is defined as $T = 1/N_\tau A_\tau$ ($N_\tau = 2, 3, 4$). Theoretically, there is no limit for N_σ in the VCE method. Effectively, because of the finite expansion order, the actual effective N_σ will be finite. The action of the SU(2) system is

$$S = \frac{\beta}{2} \tag{1}$$
$$\times \left\{ C_0 \left[\sum_{P_\sigma} \mathrm{tr}(U_{P_\sigma} + U^\dagger_{P_\sigma}) + \sum_{P_\tau} \mathrm{tr}(U_{P_\tau} + U^\dagger_{P_\tau}) \right] \right.$$

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[a] e-mail: luqiang@nankai.edu.cn

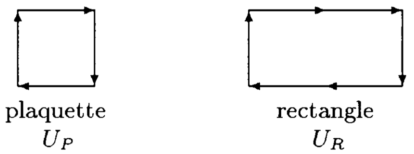


Fig. 1. Wilson loops in the improved action

$$+ \; C_1 \left[\sum_{R_\sigma} \mathrm{tr}(U_{R_\sigma} + U_{R_\sigma}^\dagger) + \sum_{R_\tau} \mathrm{tr}(U_{R_\tau} + U_{R_\tau}^\dagger) \right] \Bigg\}$$

with

$$\begin{cases} C_0 = 1, \quad C_1 = 0, \qquad \text{(Wilson action)} \\ C_0 = \frac{5}{3}, \quad C_1 = -\frac{1}{12}, \text{(Symanzik action)} \end{cases} \tag{2}$$

where U_P denotes the usual plaquette, and U_{P_σ} is an ordered product of $U_l \in \mathrm{SU}(2)$ on space-like links around a plaquette, and U_{P_τ} is an ordered product including U_l on time-like links. Similarly U_R indicates the product of link variables about a planar 2×1 rectangular loop. These terms are depicted in Fig. 1. The summations P_σ,P_τ and R_σ,R_τ are taken over all these loops; $\beta = 4/g^2$.

Consequently, the partition function is

$$Z = \int [\mathrm{d}U] \mathrm{e}^S = \mathrm{e}^{-NF}, \tag{3}$$

where $N = N_\sigma^{D-1} N_\tau$ is the total number of sites while F is the free energy per site. The periodic boundary condition of the field is

$$U^\mu(\mathbf{x}, 0) = U^\mu(\mathbf{x}, \beta). \tag{4}$$

We introduce the trial action

$$S_0 = J \sum_\sigma \mathrm{tr} U_\sigma + K \sum_\tau \mathrm{tr} U_\tau, \tag{5}$$

where U_σ and U_τ are SU(2) matrices defined on space-like and time-like links. J and K are two variational parameters which will be determined later. Correspondingly, the partition function Z_0 of the trial system is

$$\begin{aligned} Z_0 &= \int [\mathrm{d}U] \mathrm{e}^{S_0} = \left[\frac{I_1(2J)}{J} \right]^{N_s} \left[\frac{I_1(2K)}{K} \right]^{N_t} \\ &\equiv [f(J)]^{N_s} [f(K)]^{N_t}, \end{aligned} \tag{6}$$

which can be calculated exactly, where $N_s = (D-1) N_\sigma^{(D-1)} N_\tau$ and $N_t = N_\sigma^{(D-1)} N_\tau$ are the total number of space-like and time-like links, respectively.

The variational-cumulant approach takes the first step by using Z_0, instead of Z, to approach a physical quantities through a cumulant expansion. For instance,

$$\begin{aligned} Z &= \int [\mathrm{d}U] \mathrm{e}^{S-S_0} \mathrm{e}^{S_0} = Z_0 \langle \mathrm{e}^{S-S_0} \rangle_0 \\ &= Z_0 \exp \left[\sum_{n=1}^\infty \frac{1}{n!} \langle (S-S_0)^n \rangle_c \right], \end{aligned} \tag{7}$$

where

$$\langle x \rangle_0 = \int [\mathrm{d}U] x \mathrm{e}^{S_0}, \tag{8}$$

is the ordinary statistical average of x. As a consequence, any thermodynamic quantity O can be expanded and computed order by order:

$$\langle O \rangle = \langle O \rangle_0 + \sum_{n=1}^\infty \frac{1}{n!} \langle (S-S_0)^n O \rangle_c. \tag{9}$$

3 Polyakov line in variational-cumulant expansion

In the lattice space the Polyakov line is defined as

$$L(\mathbf{x}) = \frac{1}{N} \mathrm{tr} \prod_{i=1}^{N_\tau} U(\mathbf{x}, \tau_i) \tag{10}$$

and

$$\langle L(\mathbf{x}) \rangle = \mathrm{e}^{-\beta(F_q - F_0)} \begin{cases} = 0, \text{(Confinement)} \\ \neq 0, \text{(Deconfinement)} \end{cases}$$

where F_q is the free energy of the quark and anti-quark and F_0 is the energy of the vacuum. Therefore $\langle L(x) \rangle = 0$, leading to an infinite quark energy, which corresponds to the confinement phase, while $\langle L(x) \rangle \neq 0$ corresponding to a deconfinement phase. Thus, the Polyakov line can make manifest the deconfinement transition. Moreover, it is a relatively simple function to be evaluated either by the MC or VCE method. According to (9)

$$\langle L \rangle = \sum_{m=0}^\infty P_m = \langle L \rangle_0 + \sum_{m=1}^\infty \frac{1}{m} \langle (S-S_0)^m L \rangle_c. \tag{11}$$

Since $\langle L \rangle$ cannot be exactly calculated from (11), the sum must be truncated:

$$\langle L_n \rangle = \sum_{m=0}^n P_m = \langle L_0 \rangle + \sum_{m=1}^n \frac{1}{m} \langle (S-S_0)^m L \rangle_c. \tag{12}$$

We make use of an identity in [9]:

$$\begin{aligned} \langle X_1 \cdots X_n S_0^l \rangle_c = \sum_{m=0}^l C_l^m \left(J^{l-m} \frac{\partial^{l-m}}{\partial J^{l-m}} \right) \left(K^m \frac{\partial^m}{\partial K^m} \right) \\ \times \langle X_1 \cdots X_n \rangle_c. \end{aligned} \tag{13}$$

Considering that only the cumulant averages of connected diagrams are nonzero [10], the first three terms can be explicitly expressed as

$$P_0 = \langle L \rangle_0, \tag{14}$$

$$P_1 = \langle LS \rangle_c - K \frac{\partial}{\partial K} \langle L \rangle_0, \tag{15}$$

$$\begin{aligned} P_2 = \langle LS^2 \rangle_c - 2 \left[J \frac{\partial}{\partial J} + K \frac{\partial}{\partial K} \right] \langle LS \rangle_c \\ + K^2 \frac{\partial^2}{\partial K^2} \langle L \rangle_0. \end{aligned} \tag{16}$$

To abbreviate the expressions $\langle LS^n\rangle_c$ we introduce a system of diagrammatic notations. A vertical bar represents an SU(2) matrix on a time-like link, and a horizontal bar as well as an inclined bar represents a matrix on a space-like link. A closed loop means a trace of an ordered product of all matrices both on space-like and time-like links on the loop. Then we have

$$\langle L\rangle_c = \frac{1}{2}\langle L_{1,1}\rangle_0 = \langle | \rangle_0, \tag{17}$$

$$\begin{aligned}\langle LS\rangle_c &= \frac{\beta}{4}\sum_{i=1}^{3} C_{2,i}\alpha_{2,i}\langle L_{2,i}\rangle_c = \frac{\beta}{4}\Big(2rC_0\,\langle |\square\rangle_c \\ &\quad + 2rC_1\,\langle |\boxed{}\rangle_c + rC_1\,\langle |[\,]\rangle_c + rC_1\,\langle |\boxminus\rangle_c\Big).\end{aligned} \tag{18}$$

$$\begin{aligned}\langle LS^2\rangle_c &= \frac{\beta^2}{8}\sum_{i=1}^{44} C_{3,i}\alpha_{3,i}\langle L_{3,i}\rangle_c \\ &= \frac{\beta}{8}(2rC_0^2\langle L_{3,1}\rangle_c + 2rC_1^2\langle L_{3,2}\rangle_c + \cdots)\end{aligned} \tag{19}$$

In the third order expansion there are 48 inequivalent diagrams, which are listed in Table 2. $C_{n,i}$ is the product of C_0 and C_1 of the plaquette and rectangle contained in the ith diagram of the nth order. $\alpha_{n,i}$ represents the number of equivalent diagrams of the ith diagram in the nth order. We have $r = 2(D-1)$, $r_0 = r-1$, $r_i = r_{i-1} - 1$ for $i \geq 1$. $\langle x^n\rangle_c$ which is the nth order cumulant expansion average can be expressed through $\langle L_{n,i}\rangle_0$ and all $\langle L_{l,i}\rangle_c$ and $\langle D_{l,i}\rangle_c$ with $l < n$. For instance,

$$\begin{aligned}\langle L_{1,1}\rangle_c &= \langle L_{1,1}\rangle_0, \\ \langle L_{2,2}\rangle_c &= \langle L_{2,2}\rangle_0 - \langle L_{1,1}\rangle_0\langle D_{1,5}\rangle_0, \\ \langle L_{3,18}\rangle_c &= \langle L_{3,18}\rangle_0 - \langle L_{1,1}\rangle_0\langle D_{2,7}\rangle_c \\ &\quad - \langle D_{1,1}\rangle_0\langle L_{2,2}\rangle_c - \langle D_{1,5}\rangle_0\langle L_{2,1}\rangle_c \\ &\quad - \langle L_{1,1}\rangle_0\langle D_{1,1}\rangle_0\langle D_{1,5}\rangle_0.\end{aligned} \tag{20}$$

It should be noted that in spite of the periodic boundary condition, the vertical rectangle has the same contribution as that in the case with no periodic boundary condition. For instance,

$$\langle |[\,]\rangle_0 = \langle |\square\rangle_0\,. \tag{21}$$

After the truncation, $\langle L_n\rangle$ becomes dependent on the parameters J and K. A suitable set of parameters must be determined.

4 Variational treatment

To determine the variational parameters, the usual main variational approach [11] was adopted. From (3), using the standard convexity inequality $\langle e^x\rangle_0 \geq \exp\langle x\rangle_0$, one gets

$$\ln Z \geq \ln Z_0 + \langle S - S_0\rangle_0. \tag{22}$$

With $F = -\ln Z/N$, one obtains an upper bound for the free energy:

$$\begin{aligned}F \leq F_{\text{eff}} &= F_0 - \langle S - S_0\rangle_0 \\ &= -(D-1)\ln f(J) - \ln f(K) - \langle S - S_0\rangle_0.\end{aligned} \tag{23}$$

With the help of the identity (13) and only considering the connected diagrams

$$\begin{aligned}\langle S - S_0\rangle_0 &= \sum_{i=1}^{5} C_{1,i}\alpha_{1,i}Np_{1,i}\langle D_{1,i}\rangle_c - J\frac{\partial}{\partial J}\ln Z_0 \\ &= \beta\Big[C_0(D-1)A_2^2B_2^2 + \frac{1}{2}C_0(D-1)(D-2)A_2^4 \\ &\quad + C_1(D-1)A_2^4B_2^2 + C_1(D-1)(D-2)A_2^6 \\ &\quad + C_1(D-1)A_2^2B_2^4\Big] + 2(D-1)JA_2 + 2KB_2,\end{aligned} \tag{24}$$

where $A_n = I_n(2J)/I_1(2J)B_n = I_n(2K)/I_1(2K)$; $I_n(x)$ is the nth order modified Bessel function. $\alpha_{1,i} = 1$, $Np_{1,i}$ is the total number of plaquettes for the ith diagram in the first order, $\langle D_{1,i}\rangle_c$ of these diagrams are listed in Table 2. To improve the upper bound of the free energy, we choose a gauge fixing in the x direction. Then (23) and (24) are modified as

$$F_{\text{eff}} = -(D-2)\ln f(J) - \ln f(K) - \langle S - S_0\rangle_0 \tag{25}$$

and

$$\begin{aligned}\langle S - S_0\rangle_0 &= \beta\Big[C_0(D-2)A_2^2B_2^2 \\ &\quad + \frac{1}{2}C_0(D-2)(D-3)A_2^4 + C_1(D-2)A_2^4B_2^2 \\ &\quad + C_1(D-2)(D-3)A_2^6 + C_1(D-2)A_2^2B_2^4 \\ &\quad + C_0B_2^2 + C_0(D-2)A_2^2 \\ &\quad + C_1B_2^2 + C_1B_2^4 + C_1(D-2)(A_2^4 + A_2^2)\Big] \\ &\quad + 2(D-2)JA_2 + 2KB_2.\end{aligned} \tag{26}$$

To find the minimum value of (25), we arrive at two stationary conditions with respect to J and K:

$$\frac{\delta F_{\text{eff}}}{\delta J} = 0, \quad \frac{\delta F_{\text{eff}}}{\delta K} = 0, \tag{27}$$

leading to

$$\begin{aligned}J &= \beta\Big[C_0(D-3)A_2^3 \\ &\quad + C_0A_2B_2^2 + 3C_1(D-3)A_2^5 + C_0A_2 \\ &\quad + C_1(2A_2^3 + A_2) + 2C_1A_2^3B_2^2 + C_1A_2B_2^4\Big],\end{aligned} \tag{28}$$

$$\begin{aligned}K &= \beta\Big[C_0(D-2)A_2^2B_2 \\ &\quad + C_1(D-2)A_2^4B_2 + C_1B_2 \\ &\quad + C_0B_2 + 2C_1(D-2)A_2^2B_2^3 + 2C_1B_2^3\Big].\end{aligned} \tag{29}$$

In fact, the solutions of (28) and (29) are satisfied with the condition $J = K$ and $A_2 = B_2$. These two equations are the same as

$$\begin{aligned}J &= \beta\Big[C_0(D-2)A_2^3 + 3C_1(D-2)A_2^5 \\ &\quad + (C_1 + C_0)A_2 + 2C_1A_2^3\Big].\end{aligned} \tag{30}$$

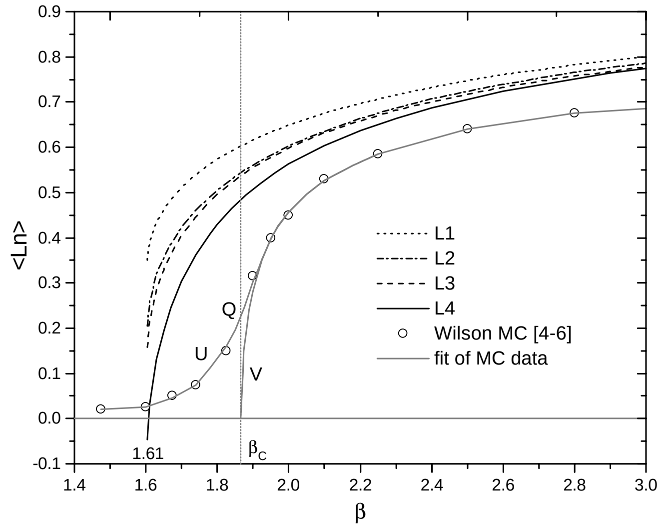


Fig. 2. $\langle L_n \rangle$, $n = 1, 2, 3, 4$, as a function of β for $N_\tau = 2$ in four dimensions for the SU(2) gauge system with the Wilson action

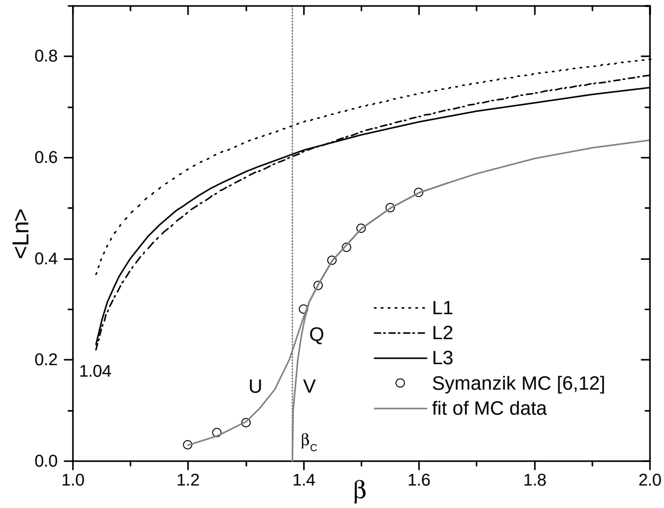


Fig. 3. $\langle L_n \rangle$, $n = 1, 2, 3$, as a function of β for $N_\tau = 2$ in four dimensions for the SU(2) gauge system with the Symanzik action

From (30) and (12), one can solve $L_n(\beta)$, the approximate Polyakov line. According to the traditional treatment of the VCE [9], the critical coupling $\beta_v = 1.61$ in Fig. 2 is determined by $\langle L_4(\beta_v) \rangle = 0$. This is a good but not satisfactory result compared with $\beta_c = 1.87$ of the MC [4–6]. To improve the accuracy of β_v, one needs to increase the expansion order. However, it is hard to expand to an order higher than the 4th due to limited machine power and the difficulty of finding the large total number of inequivalent diagrams by hand. This is the main reason for us to search for a more effective mechanism to extract the critical behavior from the first several higher order expansion terms.

For the third order expansion, (12) can be written in the form of an expansion of β:

$$\langle L_3 \rangle = a(J)\beta^2 + b(J)\beta + c(J), \tag{31}$$

where $a(J)$, $b(J)$, $c(J)$ are functions of J, For instance, $a(J) = (1/16) \sum_{i=1}^{44} \alpha_{3,i} \langle L_{3,i} \rangle_c$ for $N_\tau = 2$. We determine the critical coupling first by choosing the value of J_v which satisfies $a(J_v) = 0$. Then with the restriction of (30), the critical coupling of the VCE is

$$\beta_v = J_v \Big[C_0(D-2)A_2^3 + 3C_1(D-2)A_2^5 + (C_1 + C_0)A_2 + 2C_1 A_2^3 \Big]^{-1}. \tag{32}$$

The reason why we take this avenue is the finite size effect. Due to this effect, MC data cannot describe an ideal order parameter like the fit line V in Figs. 2 and 3. If consider the actual space size $N_\sigma = 2$ for the Wilson action, our situation is similar to that of MC (line U in Fig. 2 [5,6] and Fig. 3 [6,12]). Considering the finite size effect, one may notice that on the line U there is a point of inflection Q whose β is close to β_c.

Presuming that the VCE can reflect the behavior of the MC, we are able to determine the critical coupling through the point of inflection:

Table 1. The critical coupling β_v of the VCE (the third order) compared with the β_c of the MC (Wilson [4–6], Symanzik [6, 12]) for $N_\tau = 2, 3, 4$

N_τ	Wilson			Symanzik		
	β_v	β_c	$\lvert\beta_c - \beta_v\rvert$	β_v	β_c	$\lvert\beta_c - \beta_v\rvert$
2	2.00	1.87	0.13	1.37	1.38	0.01
3	2.40	2.20	0.20	1.63	1.60	0.03
4	2.50	2.29	0.21	1.696	1.699	0.004

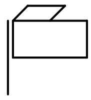
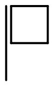
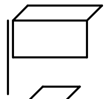
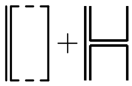
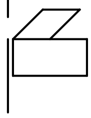
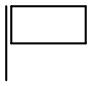
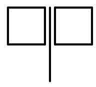
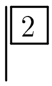
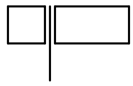
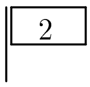
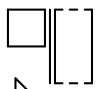
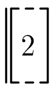
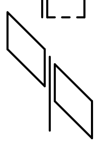
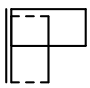
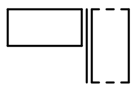
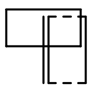
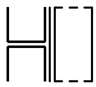
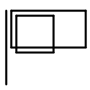
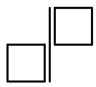
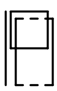
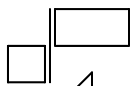
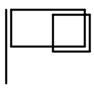
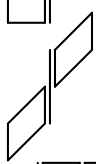
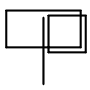
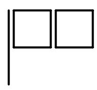
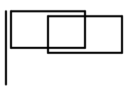
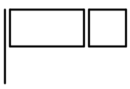
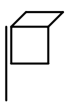
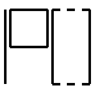
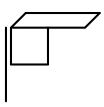
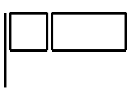
$$\frac{\partial^2 \langle L_3(\beta, J) \rangle}{\partial \beta^2} = a(J) = 0. \tag{33}$$

This restriction, which only gives the restriction of a special point of (30), will not lead to contradictions. In order to analyze $\langle L_3 \rangle$ we change (12) into an expansion in β:

$$\langle L_n \rangle = \langle L_0 \rangle + \sum_{m=1}^{n-1} \beta^m \alpha_m(J). \tag{34}$$

As n is large enough and the expansion is convergent, the finite size effect can be neglected. Given one value of J, $\langle L_n \rangle$ should give an accurate approximation of the ideal Polyakov line $\langle L \rangle$. Now our treatment is only up to the third order. Whether $\langle L_3 \rangle$ can well reflect $\langle L \rangle$ depends on the speed of convergence. Although no one has given the general proof of the converge of the cumulant expansion, the work of the SU(2) gauge model with the Wilson standard action at finite temperature [9] has yielded a good result at the fourth order expansion. Since the finite size effect cannot be neglected, it is expected that $\langle L_3 \rangle$ will reflect the variation between concavity and convexity of the line U within several expansions both for the Wilson and Symanzik action, given one value of J. That is why we use (33) to determine the critical coupling. This treatment needs the examination of final results.

Table 2. Elementary averages required for $\langle L_{n,i}\rangle_c$ in the first three terms both in the Wilson and the Symanzik action with $N_\tau = 2$

n	i	$L_{n,i}$	$\alpha_{n,i}$	$< L_{n,i} >_0$	n	i	$< L_{n,i} >$	$\alpha_{n,i}$	$< L_{n,i} >$
1	1		1	$2B_2^2$		13		$8rr_1$	$\frac{1}{2}A_2^6B_2^2(1+3A_3)(1+3B_3)$
2	1		$2r$	$A_2^2B_2^2(1+3B_3)$		14		$8rr_1$	$\frac{1}{2}A_2^6B_2^2(1+3A_3^2)(1+3B_3)$
	2		$2r$	$A_2^2B_2^2(1+3B_3)$		15		$32rr_1$	$\frac{1}{2}A_2^8B_2^2(1+3A_3)(1+3B_3)$
	3		$2r$	$A_2^4B_2^2(1+3B_3)$		16		$2rr_0$	$4A_2^4B_2^3(B_2+B_4)$
3	1		$2r$	$2B_2[B_2+A_3^2B_3(B_2+2B_4)]$		17		$2rr_0$	$4A_2^6B_2^3(B_2+B_4)$
	2		$2r$	$2B_2[B_2+A_3^4B_3(B_2+2B_4)]$		18		$4rr_0$	$4A_2^4B_2^3[B_2+B_3(B_2+B_4)]$
	3		$2r$	$2[B_2^2+A_3^2B_3^2(B_2^2+2B_4^2)]$		19		$2rr_0$	$4A_2^8B_2^3(B_2+B_4)$
	4		$8r$	$\frac{1}{2}A_2^4B_2^3[B_2(1+3B_3+3A_3+A_3B_3)$ $+8A_3B_3B_4]$		20		$8rr_0$	$4A_2^6B_2^3[B_2+B_3(B_2+B_4)]$
	5		$8r$	$\frac{1}{2}A_2^4B_2^2(1+3A_3^2)(1+3B_3^2)$		21		$4rr_0$	$4A_2^4B_2^4(B_2+B_4)$
	6		$2r$	$2A_2^2B_2^3[B_2+B_2^2(B_2+2B_4)]$		22		$2rr_0$	$\frac{1}{2}A_2^4B_2^2(1+3B_3)^2$
	7		$4r$	$\frac{1}{2}A_2^2B_2[B_2(1+3A_3+3B_3^2+A_3B_3^2)$ $+8A_3B_3B_4]$		23		$2rr_0$	$\frac{1}{2}A_2^6B_2^2(1+3B_3)^2$
	8		$2r$	$\frac{1}{2}A_2^2B_2^2(1+3B_3)(1+3B_3A_3^2)$		24		$2rr_0$	$\frac{1}{2}A_2^8B_2^2(1+3B_3)^2$
	9		$2r$	$\frac{1}{2}A_2^2B_2^2(1+3B_3)(1+3B_3A_3^2)$		25		$4rr_0$	$\frac{1}{2}A_2^4B_2^2(1+3B_3)^2$
	10		$4r$	$\frac{1}{2}A_2^4B_2^4(1+3A_3^2)(1+3B_3)$		26		$2rr_0$	$\frac{1}{2}A_2^6B_2^2(1+3B_3)^2$
	11		$8rr_1$	$\frac{1}{2}A_2^4B_2^2(1+3A_3)(1+3B_3)$		27		$4rr_0$	$\frac{1}{2}A_2^4B_2^4(1+3B_3)^2$
	12		$12rr_1$	$\frac{1}{2}A_2^6B_2^2(1+3A_3)(1+3B_3)$		28		$2rr_0$	$\frac{1}{2}A_2^6B_2^2(1+3B_3)^2$

5 Conclusions and discussions

Applying our approach to the Wilson action of the third order, with $N_\tau = 2$, we get $\beta_v = 2.0$ which is closer to that of the MC, $\beta_c = 1.87$, than the traditional value of 1.61. To make a further examination, one needs to calculate the expansion up to the fourth or fifth order. However, a high order expansion will include more complicated diagrams and will require more computer power. The Symanzik program is designed to avoid a similar situation in the MC simulation. In the VCE, to expand the Symanzik action to the third order approximates the expansion of the Wilson action to the fifth order because they have the same $N_\sigma = 4$, while the calculation of the Symanzik case is relatively simple. Using (33) on the improved Symanzik action we obtain $\beta_v = 1.37$. But the MC data give $\beta_c = 1.38$. The deviation has decreased to 0.01. Continuing this work, we calculate the cases of $N_\tau = 3, 4$ both with the Wilson and the Symanzik action. The results, which are listed in Table 1, are consistent with $N_\tau = 2$.

The accurate results of the Symanzik action make manifest that the cumulant expansion of the Polyakov line for the SU(2) gauge is convergent and the new approach to determine the critical coupling is applicable. In addition, the tree-level improved Symanzik action is more effective than the standard Wilson action in the VCE.

As a next step we plan to apply our approach to SU(3) gauge theory, which is more significant for the study of quark confinement.

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